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(Article begins on next page)

Invariant variational problems and Cartan connections on gauge-natural bundles

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Abstract

A principal Cartan connection is canonically defined by gauge-natural invariant variational problems of finite order due to the existence of a reductive split structure associated with canonical Lagrangian conserved quantities on gauge-natural bundles.

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Key words: jet space; variational sequence; reductive structure; Cartan connection

1 Introduction

In the following classical physical fields are assumed to be sections of bundles functorially associated with gauge-natural prolongations of principal bundles [2, 5]. We consider finite order Lagrangian variational problems in terms of exterior differentials of forms modulo contact forms as framed in the context of finite order variational sequences [7].

As well known, following Noether's theory [8], from invariance properties of the Lagrangian the existence of suitable conserved currents and identities can be deduced. Within such a picture *generalized Bergmann–Bianchi identities* are conditions for a Noether conserved current to be not only closed but also the global divergence of a tensor density called a superpotential [10]. Recently, we proposed an approach to deal with the problem of *canonical* covariance and uniqueness of conserved quantities which uses *variational*

derivatives taken with respect to the class of (generalized) variation vector fields being Lie derivatives of sections of bundles by gauge-natural lifts of infinitesimal principal automorphisms [9, 11, 12, 13, 3].

In this note, we shortly review some of the outcomes and in particular we recall how the kernel of the gauge-natural Jacobi morphism (coinciding with generalized Bergmann–Bianchi identities) defines a split reductive structure on the relevant underlying principal bundle. As a consequence, we prove that a principal Cartan connection is canonically defined by gauge-natural invariant variational problems of finite order.

Let us recall some useful concepts of prolongations, mainly with the aim of fixing the notation; for details see *e.g.* [5, 14]. Let $\pi : \mathbf{Y} \rightarrow \mathbf{X}$ be a fibered manifold, with $\dim \mathbf{X} = n$ and $\dim \mathbf{Y} = n + m$. For $s \geq q \geq 0$ integers we deal with the s -jet space $J_s \mathbf{Y}$ of s -jet prolongations of (local) sections of π ; in particular, we set $J_0 \mathbf{Y} \equiv \mathbf{Y}$. We recall that there are the natural fiberings $\pi_q^s : J_s \mathbf{Y} \rightarrow J_q \mathbf{Y}$, $s \geq q$, $\pi^s : J_s \mathbf{Y} \rightarrow \mathbf{X}$, and, among these, the *affine* fiberings π_{s-1}^s . We denote by $V\mathbf{Y}$ the vector subbundle of the tangent bundle $T\mathbf{Y}$ of vectors on \mathbf{Y} which are vertical with respect to the fibering π .

For $s \geq 1$, we consider the following natural splitting induced by the natural contact structure on jets bundles (see *e.g.* [6, 7]): $J_s \mathbf{Y} \times_{J_{s-1} \mathbf{Y}} T^* J_{s-1} \mathbf{Y} = J_s \mathbf{Y} \times_{J_{s-1} \mathbf{Y}} (T^* \mathbf{X} \oplus V^* J_{s-1} \mathbf{Y})$.

A vector field ξ on \mathbf{Y} is said to be *vertical* if it takes values in $V\mathbf{Y}$. A vertical vector field can be prolonged to a vertical vector field $j_s \xi$ characterized by the fact that its flow is the natural prolongation of the flow of ξ . Given a vector field $\Xi : J_s \mathbf{Y} \rightarrow T J_s \mathbf{Y}$, the above splitting yields $\Xi \circ \pi_s^{s+1} = \Xi_H + \Xi_V$, where Ξ_H and Ξ_V are the horizontal and the vertical part of Ξ , respectively. As well known, the above splitting induces also a decomposition of the exterior differential on \mathbf{Y} , $(\pi_r^{r+1})^* \circ d = d_H + d_V$, where d_H and d_V are called the *horizontal* and *vertical differential*, respectively. Such decompositions always rise the order of the objects.

Let $\mathbf{P} \rightarrow \mathbf{X}$ be a principal bundle with structure group \mathbf{G} . For $r \leq k$ integers consider the *gauge-natural prolongation* of \mathbf{P} given by $\mathbf{W}^{(r,k)} \mathbf{P} \doteq J_r \mathbf{P} \times_{\mathbf{X}} L_k(\mathbf{X})$, where $L_k(\mathbf{X})$ is the bundle of k -frames in \mathbf{X} [2, 5]; $\mathbf{W}^{(r,k)} \mathbf{P}$ is a principal bundle over \mathbf{X} with structure group $\mathbf{W}_n^{(r,k)} \mathbf{G}$ which is the *semidirect* product with respect to the action of $GL_k(n)$ on \mathbf{G}_n^r given by jet composition and $GL_k(n)$ is the group of k -frames in \mathbb{R}^n . Here we denote by \mathbf{G}_n^r the space of (r, n) -velocities on \mathbf{G} .

Let \mathbf{F} be a manifold and $\zeta : \mathbf{W}_n^{(r,k)} \mathbf{G} \times \mathbf{F} \rightarrow \mathbf{F}$ be a left action of $\mathbf{W}_n^{(r,k)} \mathbf{G}$ on \mathbf{F} . There is a naturally defined right action of $\mathbf{W}_n^{(r,k)} \mathbf{G}$ on $\mathbf{W}^{(r,k)} \mathbf{P} \times \mathbf{F}$ so that we have in the standard way the associated *gauge-natural bundle* of order (r, k) : $\mathbf{Y}_\zeta \doteq \mathbf{W}^{(r,k)} \mathbf{P} \times_\zeta \mathbf{F}$. All our considerations shall refer to \mathbf{Y} as a gauge-natural bundle as just defined.

Denote now by $\mathcal{A}^{(r,k)}$ the sheaf of right invariant vector fields on $\mathbf{W}^{(r,k)} \mathbf{P}$. The *gauge-natural lift* is defined as the functorial map $\mathfrak{G} : \mathbf{Y}_\zeta \times_{\mathbf{X}} \mathcal{A}^{(r,k)} \rightarrow T\mathbf{Y}_\zeta : (\mathbf{y}, \bar{\Xi}) \mapsto \hat{\Xi}(\mathbf{y})$, where, for any $\mathbf{y} \in \mathbf{Y}_\zeta$, one sets: $\hat{\Xi}(\mathbf{y}) = \frac{d}{dt}[(\Phi_{\zeta t})(\mathbf{y})]_{t=0}$, and $\Phi_{\zeta t}$ denotes the (local) flow corresponding to the gauge-natural lift of Φ_t . Such a functor defines a class of parametrized contact transformations.

This mapping fulfils the following properties (see [5]): \mathfrak{G} is linear over $id_{\mathbf{Y}_\zeta}$; we have $T\pi_\zeta \circ \mathfrak{G} = id_{T\mathbf{X}} \circ \bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $\mathbf{Y}_\zeta \times_{\mathbf{X}} \mathcal{A}^{(r,k)} \rightarrow T\mathbf{X}$; for any pair $(\bar{\Lambda}, \bar{\Xi}) \in \mathcal{A}^{(r,k)}$, we have $\mathfrak{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}), \mathfrak{G}(\bar{\Xi})]$. In the following, by an abuse of notation we denote by $V\mathcal{A}^{(r,k)}$ the bundle of vertical parts of sections of $\mathcal{A}^{(r,k)} \rightarrow \mathbf{X}$.

Let γ be a (local) section of \mathbf{Y}_ζ , $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [5] we define the *generalized Lie derivative* of γ along the vector field $\hat{\Xi}$ to be the (local) section $\mathcal{L}_{\bar{\Xi}}\gamma : \mathbf{X} \rightarrow V\mathbf{Y}_\zeta$, given by $\mathcal{L}_{\bar{\Xi}}\gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma$.

The Lie derivative operator acting on sections of gauge-natural bundles is an homomorphism of Lie algebras; furthermore, for any gauge-natural lift, the fundamental relation holds true: $\hat{\Xi}_V = -\mathcal{L}_{\bar{\Xi}}$.

2 Gauge-natural Jacobi equations and Cartan connections

The fibered splitting induced by the contact structure on finite order jets yields the *sheaf splitting* $\mathcal{H}_{(s+1,s)}^p = \bigoplus_{t=0}^p \mathcal{C}_{(s+1,s)}^{p-t} \wedge \mathcal{H}_{s+1}^t$, where the sheaves $\mathcal{H}_{(s,q)}^p$ and \mathcal{H}_s^p of *horizontal forms* with respect to the projections π_q^s and π_0^s , respectively, while $\mathcal{C}_{(s,q)}^p \subset \mathcal{H}_{(s,q)}^p$ and $\mathcal{C}_s^p \subset \mathcal{C}_{(s+1,s)}^p$ are *contact forms*, i.e. horizontal forms valued into $\mathcal{C}_s^*[Y]$ (they have the property of vanishing along any section of the gauge-natural bundle). We put $\mathcal{H}_{s+1}^{p,h} \doteq h(\Lambda_s^p)$ for $0 < p \leq n$ and the map h is the *horizontalization*, i.e. the projection on the summand of lesser contact degree. Let $\eta \in \mathcal{C}_s^1 \wedge \mathcal{C}_{(s,0)}^1 \wedge \mathcal{H}_{s+1}^{n,h}$; then there is a unique morphism $K_\eta \in \mathcal{C}_{(2s,s)}^1 \otimes \mathcal{C}_{(2s,0)}^1 \wedge \mathcal{H}_{2s+1}^{n,h}$ such that, for all $\Xi : \mathbf{Y} \rightarrow V\mathbf{Y}$, $C_1^1(j_{2s}\Xi \otimes K_\eta) = E_{j_s\Xi}|_\eta$, where C_1^1 stands for tensor contraction on the first

factor and \rfloor denotes inner product and $E_{j_s \Xi \rfloor \eta} = (\pi_{s+1}^{2s+1})^* j_s \Xi \rfloor \eta + F_{j_s \Xi \rfloor \eta}$ (with $F_{j_s \Xi \rfloor \eta}$ a local divergence) is a uniquely defined global section of $\mathcal{C}_{(2s,0)}^1 \wedge \mathcal{H}_{2s+1}^{n,h}$ (see [15]).

By an abuse of notation, let us denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$ in the standard way. We set $\Theta_s^* \doteq \ker h + d \ker h$. We have that $0 \rightarrow \mathbb{R}_Y \rightarrow \mathcal{V}_s^*$, where $\mathcal{V}_s^* = \Lambda_s^* / \Theta_s^*$, is an exact resolution of the constant sheaf \mathbb{R}_Y [7]. A section $E_{d\lambda} \doteq \mathcal{E}_n(\lambda) \in \mathcal{V}_s^{n+1}$ is the *generalized higher order Euler–Lagrange type morphism* associated with λ .

The morphism K_η can be integrated by parts to provide a representation of the *generalized Jacobi morphism* associated with λ [10]. Let λ be a Lagrangian and consider $\hat{\Xi}_V$ as a variation vector field. Let us set $\chi(\lambda, \hat{\Xi}_V) \equiv E_{j_s \hat{\Xi} \rfloor h d \mathcal{L}_{j_{2s+1} \hat{\Xi}_V} \lambda}$. Because of linearity properties of $K_{h d \mathcal{L}_{j_{2s} \hat{\Xi}_V} \lambda}$, and by using a global decomposition formula due to Kolář, we can decompose the morphism defined above as $\chi(\lambda, \hat{\Xi}_V) = E_{\chi(\lambda, \hat{\Xi}_V)} + F_{\chi(\lambda, \hat{\Xi}_V)}$, where $F_{\chi(\lambda, \hat{\Xi}_V)}$ is a *local* horizontal differential which can be globalized by fixing of a connection.

Definition 1 We call the morphism $\mathcal{J}(\lambda, \hat{\Xi}_V) \doteq E_{\chi(\lambda, \hat{\Xi}_V)}$ the *gauge-natural generalized Jacobi morphism* associated with the Lagrangian λ and the variation vector field $\hat{\Xi}_V$. We call the morphism $\mathfrak{H}(\lambda, \hat{\Xi}_V) \doteq \hat{\Xi}_V \rfloor \mathcal{E}_n(\hat{\Xi}_V \rfloor \mathcal{E}_n(\lambda))$ the *gauge-natural Hessian morphism* associated with λ . \square

The morphism $\mathcal{J}(\lambda, \hat{\Xi}_V)$ is a *linear* morphism with respect to the projection $J_{4s} \mathbf{Y}_\zeta \times_V J_{4s} \mathcal{A}^{(r,k)} \rightarrow J_{4s} \mathbf{Y}_\zeta$. Notice that, since $\delta_{\mathfrak{g}}^2 \lambda \doteq \mathcal{L}_{\hat{\Xi}_V} \mathcal{L}_{\hat{\Xi}_V} \lambda = \hat{\Xi}_V \rfloor \mathcal{E}_n(\hat{\Xi}_V \rfloor \mathcal{E}_n(\lambda))$, we have $\mathfrak{H}(\lambda, \hat{\Xi}_V) = \delta_{\mathfrak{g}}^2 \lambda$; furthermore, being also $\delta_{\mathfrak{g}}^2 \lambda = \mathcal{E}_n(\hat{\Xi}_V \rfloor h(d\delta \lambda))$ [10], then $\mathfrak{H}(\lambda, \hat{\Xi}_V)$ is self-adjoint. Furthermore, we have $\mathcal{J}(\lambda, \hat{\Xi}_V) \doteq E_{\chi(\lambda, \hat{\Xi}_V)} = \mathcal{E}_n(\hat{\Xi}_V \rfloor h(d\delta \lambda)) = \mathfrak{H}(\lambda, \hat{\Xi}_V)$, stating that *the Hessian and as a consequence also the Jacobi morphism are symmetric self-adjoint morphisms*. The Jacobi morphism $\mathcal{J}(\lambda, \hat{\Xi}_V)$ can be interpreted as an endomorphism of $J_{4s} V \mathcal{A}^{(r,k)}$.

In the following we concentrate on some geometric aspects of the space $\mathfrak{K} \doteq \ker \mathcal{J}(\lambda, \hat{\Xi}_V)$. Such a kernel defines generalized gauge-natural Jacobi equations [10], the solutions of which we call *generalized Jacobi vector fields*. It characterizes *canonical* covariant conserved quantities. In fact, given $[\alpha] \in \mathcal{V}_s^n$, since the *variational Lie derivative* of classes of forms can be represented the variational sequence, we have the corresponding version of the First Noether Theorem:

$$\mathcal{L}_{j_s \Xi} [\alpha] = \omega(\lambda, \hat{\Xi}_V) + d_H(j_{2s} \hat{\Xi}_V \rfloor p_{d_V h(\alpha)} + \xi \rfloor h(\alpha)), \quad (1)$$

where we put $\omega(\lambda, \hat{\Xi}_V) \doteq \hat{\Xi}_V \rfloor \mathcal{E}_n(\lambda) \doteq -\mathcal{L}_{\hat{\Xi}} \rfloor \mathcal{E}_n(\lambda)$.

As usual, λ is defined a *gauge-natural invariant Lagrangian* if the gauge-natural lift $(\hat{\Xi}, \xi)$ of *any* vector field $\Xi \in \mathcal{A}^{(r,k)}$ is a symmetry for λ , *i.e.* if $\mathcal{L}_{j_{s+1}\hat{\Xi}} \lambda = 0$. In this case, as an immediate consequence we have that $\omega(\lambda, \hat{\Xi}_V) = d_H(-j_s \mathcal{L}_{\hat{\Xi}} \rfloor p_{d_V} \lambda + \xi \rfloor \lambda)$.

The generalized *Bergmann–Bianchi morphism* $\beta(\lambda, \hat{\Xi}_V) \doteq E_{\omega(\lambda, \hat{\Xi}_V)}$ is canonically vanishing along \mathfrak{K} . This fact characterizes canonical covariant conserved Noether currents [10, 9]. Furthermore, along the kernel of the gauge-natural generalized gauge-natural Jacobi morphism we have that $\mathcal{L}_{j_{s+1}\hat{\Xi}_H} [\mathcal{L}_{j_{s+1}\hat{\Xi}_V} \lambda] \equiv 0$. Hence Bergmann–Bianchi identities are equivalent to the invariance condition $\mathcal{L}_{j_{s+1}\hat{\Xi}} [\mathcal{L}_{j_{s+1}\hat{\Xi}_V} \lambda] \equiv 0$ and can be suitably interpreted as Noether identities associated with the invariance properties of the Euler–Lagrange morphism $\mathcal{E}_n(\omega)$ [12]. This fact can be used to prove that \mathfrak{K} is characterized as a vector subbundle, being the kernel of a Hamiltonian operator [3, 11].

Proposition 1 *A principal Cartan connection is canonically defined by gauge-natural invariant variational problems of finite order.*

PROOF. Let \mathfrak{h} be the Lie algebra of right-invariant vertical vector fields on $W^{(r+4s, k+4s)} \mathbf{P}$ and \mathfrak{k} the Lie subalgebra of generalized Jacobi vector fields defined as solutions of generalized Jacobi equations. Consider now that, since the Jacobi morphism self-adjoint, its cokernel coincides with the cokernel of the adjoint morphism, thus we have that $\dim \mathfrak{K} = \dim \text{Coker } \mathcal{J}$. If we further consider that \mathfrak{K} is of constant rank because it is the kernel of a Hamiltonian operator [11], we are able to define the split structure given by $\mathfrak{h} = \mathfrak{K} \oplus \text{Im } \mathcal{J}$. The Lie derivative of a solution of Euler–Lagrange equations *with respect to a Jacobi vector field* is again a solution of Euler–Lagrange equations. However, the Lie derivative with respect to vertical parts of the commutator between the gauge-natural lift of a Jacobi vector field and (the vertical part of) a lift not lying in \mathfrak{K} *is not* a solution of Euler–Lagrange equations. Thus, since \mathcal{J} is a projector and a derivation of \mathfrak{h} , it is easy to see that the split structure is also reductive, being $[\mathfrak{k}, \text{Im } \mathcal{J}] = \text{Im } \mathcal{J}$. We have then proved that *the kernel \mathfrak{K} defines a reductive structure on $W^{(r+4s, k+4s)} \mathbf{P}$.*

In particular, for each $\mathbf{p} \in W^{(r,k)} \mathbf{P}$ by denoting $\mathcal{W} \equiv \mathfrak{h}_{\mathbf{p}}$, $\mathcal{K} \equiv \mathfrak{k}_{\mathbf{p}}$ and $\mathcal{V} \equiv \text{Im } \mathcal{J}_{\mathbf{p}}$ we have the reductive Lie algebra decomposition $\mathcal{W} = \mathcal{K} \oplus \mathcal{V}$, with $[\mathcal{K}, \mathcal{V}] = \mathcal{V}$. Notice that \mathcal{W} is the Lie algebra of the Lie group $W_n^{(r,k)} \mathbf{G}$. Since \mathfrak{K} is a vector subbundle of $\mathcal{A}^{(r,k)} = TW^{(r,k)} \mathbf{P} / W_n^{(r,k)} \mathbf{G}$ there exists a principal subbundle $\mathbf{Q} \subset W^{(r,k)} \mathbf{P}$ such that $\dim \mathbf{Q} = \dim \mathcal{W}$, $\mathcal{K} = T\mathbf{Q}/K|_{\mathbf{q}}$,

where \mathbf{K} is the (reduced) Lie group of the Lie algebra \mathcal{K} and the embedding $\mathbf{Q} \rightarrow \mathbf{W}^{(r,k)}\mathbf{P}$ is a principal bundle homomorphism over the injective group homomorphism $\mathbf{K} \rightarrow \mathbf{W}_n^{(r,k)}\mathbf{G}$.

Now, if ω is a principal connection on $\mathbf{W}^{(r,k)}\mathbf{P}$, the restriction $\omega|_{\mathbf{Q}}$ is a Cartan connection of the principal bundle $\mathbf{Q} \rightarrow \mathbf{X}$. In fact, let us consider a principal connection $\bar{\omega}$ on the principal bundle \mathbf{Q} *i.e.* a \mathcal{K} -invariant horizontal distribution defining the vertical parallelism $\bar{\omega} : V\mathbf{Q} \rightarrow \mathcal{K}$ by means of the fundamental vector field mapping in the usual and standard way. Since \mathcal{K} is a subalgebra of the Lie algebra \mathcal{W} and $\dim \mathbf{Q} = \dim \mathcal{W}$, it is defined a principal Cartan connection of type \mathcal{W}/\mathcal{K} , that is an \mathcal{W} -valued absolute parallelism $\hat{\omega} : T\mathbf{Q} \rightarrow \mathcal{W}$ which is an homomorphism of Lie algebras, when restricted to \mathcal{K} , preserving Lie brackets if one of the arguments is in \mathcal{K} , and such that $\hat{\omega}|_{V\mathbf{Q}} = \bar{\omega}$, that means that $\hat{\omega}$ is an extension of the natural vertical parallelism.

We have then to show that such a $\hat{\omega}$ exists. We can define $\hat{\omega}$ as the restriction of the natural vertical parallelism defined by a principal connection ω on $\mathbf{W}^{(r,k)}\mathbf{P}$ by means of the fundamental vector field mapping $\omega : V\mathbf{W}^{(r,k)}\mathbf{P} \rightarrow \mathcal{W}$ to $T\mathbf{Q}$. This restriction is, in particular, \mathcal{K} -invariant since is by construction \mathcal{W} -invariant. Of course, this definition is well done provided that $T\mathbf{Q} \subset V\mathbf{W}^{(r,k)}\mathbf{P}$. In fact, it is easy to see that $T\mathbf{Q} \subset V\mathbf{W}^{(r,k)}\mathbf{P}$ holds true as a consequence of the reductive split structure on $\mathbf{W}^{(r+4s,k+4s)}\mathbf{P}$. In particular, $\forall \mathbf{q} \in \mathbf{Q}$, we have $T_{\mathbf{q}}\mathbf{Q} \cap \mathcal{H}_{\mathbf{q}} = 0$, where $\mathcal{H}_{\mathbf{q}}, \forall \mathbf{p} \in \mathbf{W}^{(r,k)}\mathbf{P}$ is defined by ω as $T_{\mathbf{p}}\mathbf{W}^{(r,k)}\mathbf{P} = V_{\mathbf{p}}\mathbf{W}^{(r,k)}\mathbf{P} \oplus \mathcal{H}_{\mathbf{p}}$; furthermore, $\dim \mathbf{X} = \dim \mathcal{W}/\mathcal{K}$.

□ QED

Let us now explicate some consequences. In fact, let us consider once more the reductive decomposition $\mathcal{W} = \mathcal{K} \oplus \mathcal{V}$, with $[\mathcal{K}, \mathcal{V}] = \mathcal{V}$. The \mathcal{K} -component $\eta = pr_{\mathcal{K}} \circ \hat{\omega}$ is a principal connection form on the \mathcal{K} -manifold \mathbf{Q} . A \mathcal{K} -invariant horizontal distribution $\mathfrak{H} = \hat{\omega}^{-1}(\mathcal{V}) \subset \mathbf{Q}$ complementary to the \mathcal{K} -invariant vertical distribution $\zeta_{\mathcal{K}}(\mathbf{Q}) \subset \mathbf{Q}$ spanned by the \mathcal{K} -action and such that $[\zeta_{\mathcal{K}}, \Gamma(\mathfrak{H})] \subset \Gamma(\mathfrak{H})$, with $\Gamma(\mathfrak{H}) \subset \chi(\mathbf{Q})$ is the space of section of the bundle \mathfrak{H} . The \mathcal{V} -component $\theta = pr_{\mathcal{V}} \circ \hat{\omega}$ is a sort of a displacement form and we have $\ker \theta = \zeta_{\mathcal{K}}(\mathbf{Q})$. In fact, being \mathbf{K} a reductive Lie subgroup of $\mathbf{W}_n^{(r,k)}\mathbf{G}$ the principal Cartan connection could be seen as a \mathbf{K} -structure equipped with a principal connection form η on \mathbf{Q} . By considering the reduction of the structure bundle $\mathbf{W}^{(r,k)}\mathbf{P}$ to a subbundle with structure group a subgroup of the differential group (of a certain order), we see that generalized Jacobi vector fields can be interpreted as a kind of reductive gauge-natural lift in

the sense of [4].

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